University of California at Irvine
MAE106 Mechanical Systems Laboratory
Time and Frequency Domains

Conceptual overview:

1. Why do engineers analyze systems in both the time and frequency domain?

Why the time domain? We ________ in the time domain.

Typical questions:
- How does the system respond to a ________ input? Example:
- How does the system respond to a ________ input? Example:
- How fast does the system respond? (Useful #: ____________)
- Does it __________ ?
- Does it __________ ?

Why the frequency domain?

a. Intuition

   Systems act like ________ , responding differently to inputs at different frequencies

   Four common types of filters:

b. Ease – sometimes its easier to solve differential equations in the frequency domain (Laplace Transform)

2. What is a transfer function and what is a frequency response?

A linear differential equation in the time domain becomes a transfer function in the ______________.

To see this take the Laplace transform of a differential equation:

FACT: The transfer function tells how a system responds to any input in the frequency domain. The output is just the input multiplied by the ___________________.

The transfer function also tells how a system responds to a sinusoidal input.

FACT: Using Laplace Transforms, it is possible to prove that: sine wave in $\Rightarrow$

The transfer function tells how much an input sine wave is ____________ and ___________as a function of its frequency.

Knowing these two things means you know the _________________ of the system, which is characterized by the and the _________________ .

These facts are very useful when combined with two other facts:

FACT: Any signal can be represented as the sum of ____________.

FACT: The response of a linear system to the sum of two inputs is the sum of the____________________.

THESE FACTS LET US THINK OF LINEAR SYSTEMS AS FILTERS:

Important Frequency Domain Concepts

Why study systems in the Frequency Domain?

1) Helps us develop intuition about how linear time-invariant systems "process" their inputs. The key picture to have in your mind is that systems behave like filters. In particular:
   a) a sinusoidal input produces a sinusoidal output that is scaled and shifted, but at the same frequency as the input. Scaling and shifting is described by the systems "frequency response".
   b) any signal can viewed as a sum of sinusoids
   c) Thus, we can think of a linear system as "breaking down" its input signal into a sum of sinusoids; processing each sinusoid according to its "frequency response"; then reassembling the processed individual frequencies into the output signal. In this framework, we can view systems as filters such as "low-pass", "high-pass", "band-pass", and "notch" filters.
2) Makes solving for the system response easier.

Laplace transform: a mathematical operation performed on a signal or system that transforms it into the complex frequency domain.

Transfer function: A complex function that relates a system's input to its output in the complex frequency (or "Laplace") domain. Found by taking the Laplace transform of the differential equation that describes the system. Assumes system is stable and response to initial conditions has died out (i.e. system is in the "steady state").

Impedance: A special case of a transfer function, which relates current to voltage for circuit elements. Allows us to do circuit analysis on inductors and capacitors as if they are resistors (although the value of the equivalent "R" varies with the frequency of the current going into the capacitor and inductor).

Frequency response: A set of two functions that describe how a system scales and shifts a sinusoidal input. The two functions are the "magnitude response" (describes scaling, also called "amplitude response") and the "phase response" (describes phase shifting). Note that the frequency response of a system can conveniently be described by a complex function which is the transfer function with $j\omega$ substituted for s (see proof in class notes).

Stability: a property of a system in which bounded inputs produce bounded outputs (i.e. system output does not "blow up").

Characteristic polynomial: The denominator of the transfer function; determines system's stability.

Poles: the roots of the characteristic polynomial. Poles are usually complex numbers. If the real part of any of the poles is greater than zero (i.e. the pole lies in the right half of the complex plane), then the system is unstable. You can prove this by taking the inverse Laplace transform of the transfer function – poles in the right half plane inverse transform to exponentials in time that "blow up".

Zeros: the roots of the numerator of a transfer function.

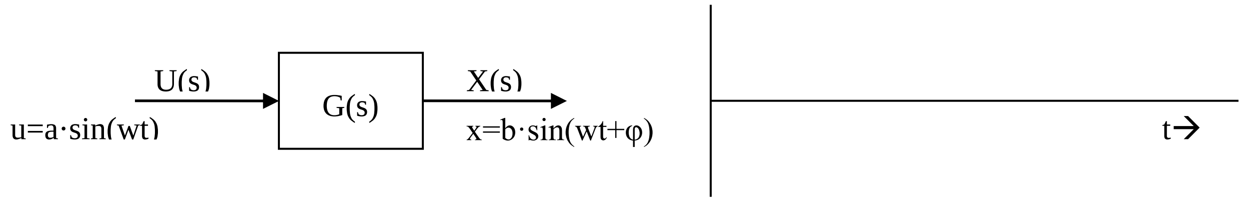
Partial fraction expansion: a method for finding the inverse Laplace transform of a signal or system. The method involves expanding a transfer function with a high-order characteristic polynomial (and an unknown inverse Laplace transform) into a sum of lower-order transfer functions (for which the inverse Laplace transforms are known).

Bode plot: a common way to plot the frequency response of a system. A Bode plot has two plots. The first plot is the "magnitude response plot" that shows how the system scales a sinusoidal input as a function of its frequency. A Bode plot shows the magnitude response by plotting the scaling factor in decibels versus ω on a log scale. The second plot is the "phase response plot" that shows how the system phase-shifts a sinusoidal input as a function of its frequency. A Bode plot shows the phase response by plotting the phase shift that the system introduces into its input versus ω on a log scale. The reason Bode plots are popular is because they're easy to hand-draw: the magnitude response plot looks approximately like connected line segments on a log scale. Because they are easy to draw, they are also easy to interpret.

Decibel: a unit of measurement of a scaling factor A. #dB = 20 log A

Proof that sine wave input gives a sine wave output with different amplitude and phase

Overview:
If we input a sine wave to a stable linear system, the output will be a ________ wave of the same frequency with different ________________ and ________.



For the signals shown x(t) _______ u(t) by φ°, and this means that φ___0.

Mathematical model of the system:
How can we predict the output amplitude and phase? Consider a general n^th order linear system:

$$\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_0 x = b_{n-1}\frac{d^{n-1}u}{dt^{n-1}} + b_{n-2}\frac{d^{n-2}x}{dt^{n-2}} + \ldots + b_0 u$$

If we take the Laplace Transform:

$$X(\underbrace{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}_{A(s)}) = U(\underbrace{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}_{B(s)}) + IC(s) \rightarrow$$

A(s) is called the _______________________of the system, and we can write it in factored form:

What happen if U(s)=0?

$$X = \frac{IC(s)}{A(s)} \quad \rightarrow$$

The system is stable if Re(s_i)__ 0, where s_i are the ______ of the _____________________ A(s) .
Assume we have a stable system, and we apply a sine wave to the system:

$$u(t) = a \cdot \sin wt \quad \rightarrow$$

Since A(s) is stable, the contribution of the response due to the initial conditions decays to zero.
Use partial fraction expansion to find the inverse Laplace transform.

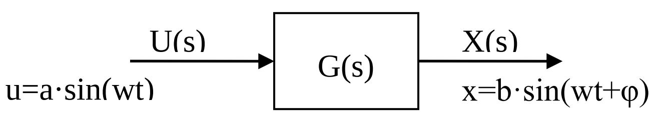
The inverse Laplace transform has the form: (Trick: $L\left[e^{-at}\right] = \dfrac{1}{s+a}$)

The terms $e^{s_i t}$ decay to zero, because $\mathrm{Re}(s_i)$ ___ 0. Use partial fraction expansion to find K_1 and K_2.

We can write $G(jw)$ in polar coordinates as: $G(jw) = |G(jw)| e^{j\varphi}$, where $\varphi = \angle G(jw)$. So that we can express $G(-jw)$ as: (hint: $G(-jw)$ is the complex conjugate of $G(jw)$).

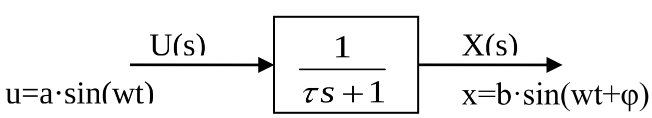
But using Euler's formula: $\sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$, we get an output of the form:

Summary:
If we input a sine wave of amplitude *a* to a stable linear system, the output will be a ________ wave of the same frequency with different __________ and ________.



The output amplitude *b* can be computed as:
$b = a \cdot |G(jw)|$
The output phase can be computed as:
$\varphi = \angle G(jw)$

Example:



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